Mutation Invariance of the Szabó Spectral Sequence

Aayush Karan
Mentor: Dr. Jianfeng Lin

May 19, 2018

MIT PRIMES Conference

Preliminary Knot Theory

- A *knot* can be conceptualized as a piece of string looped in and out of itself with the ends fused together. Multiple disjoint knots form *links*.

Preliminary Knot Theory

- A *knot* can be conceptualized as a piece of string looped in and out of itself with the ends fused together. Multiple disjoint knots form *links*.
- The projection of a link onto the plane is its *link diagram* $\mathcal{D}$, and the image of two overlapping strands is a *crossing*.

Preliminary Knot Theory

- A *knot* can be conceptualized as a piece of string looped in and out of itself with the ends fused together. Multiple disjoint knots form *links*.
- The projection of a link onto the plane is its *link diagram* $\mathcal{D}$, and the image of two overlapping strands is a *crossing*.
- The purpose of link transformations is to see what stays the same after altering the link. *Link invariants* are “canonical equivalences.”

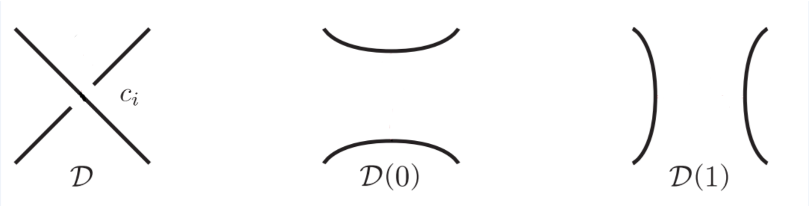
Resolution Diagrams

Given a link diagram $\mathcal{D}$, we can enumerate the crossings $c_1, c_2, \ldots, c_n$, and *smooth* them:

Resolution Diagrams

Given a link diagram $\mathcal{D}$, we can enumerate the crossings $c_1, c_2, \ldots, c_n$, and *smooth* them:

- There are two types of *resolutions*: 0-resolutions and 1-resolutions
- We get a binary string α associated with $\mathcal{I}_\alpha$, a complete smoothing of $\mathcal{D}$.



Resolution Cube

Let the *weight* of $\mathcal{I}_\alpha$, denoted $|\mathcal{I}_\alpha|$, be the number of 1's in α.

Resolution Cube

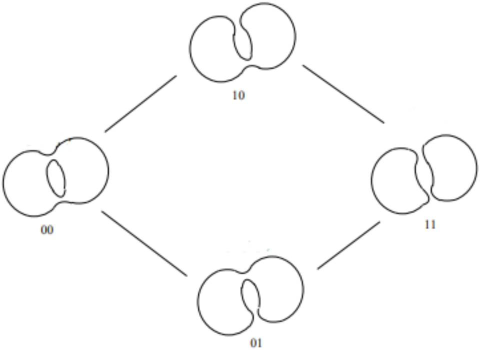
Let the *weight* of $\mathcal{I}_\alpha$, denoted $|\mathcal{I}_\alpha|$, be the number of 1's in α.

- A helpful way to visualize all the resolution diagrams is to create a *resolution cube*, where we arrange the $\mathcal{I}_\alpha$ column-wise grouped by weight.

Resolution Cube

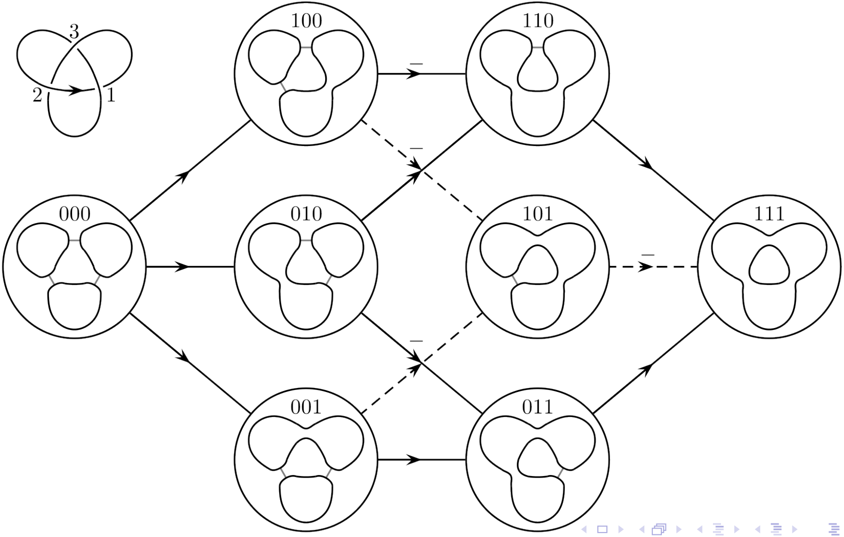
Let the *weight* of $\mathcal{I}_\alpha$, denoted $|\mathcal{I}_\alpha|$, be the number of 1's in α.

- A helpful way to visualize all the resolution diagrams is to create a *resolution cube*, where we arrange the $\mathcal{I}_\alpha$ column-wise grouped by weight.
- Example:



Resolution Cube

Another Example:



Resolution Cube

Note that there are edges in the resolution cube.

- These are known as *1-dimensional faces*, with diagrams $\mathcal{I}_\alpha$ and $\mathcal{I}_{\alpha'}$ adjacent iff α and α' differ at precisely one position.

Resolution Cube

Note that there are edges in the resolution cube.

- These are known as *1-dimensional faces*, with diagrams $\mathcal{I}_\alpha$ and $\mathcal{I}_{\alpha'}$ adjacent iff α and α' differ at precisely one position.
- We can also consider *k-dimensional faces*, such that α and α' differ at precisely k positions.

A Special Bargain

Homological Algebra

- "Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine" — Sir Michael Francis Atiyah

Definition (Chain Complex)

A *chain complex* (M_*, d) with *differential* d is a sequence of homomorphisms between vector spaces

$$\ldots \xrightarrow{d_{n-2}} M^{n-1} \xrightarrow{d_{n-1}} M^n \xrightarrow{d_n} M^{n+1} \xrightarrow{d_{n+1}} \ldots$$

such that $d_{n+1} \circ d_n = 0$ for each n.

Definition (Chain Complex)

A *chain complex* (M_*, d) with *differential* d is a sequence of homomorphisms between vector spaces

$$\ldots \xrightarrow{d_{n-2}} M^{n-1} \xrightarrow{d_{n-1}} M^n \xrightarrow{d_n} M^{n+1} \xrightarrow{d_{n+1}} \ldots$$

such that $d_{n+1} \circ d_n = 0$ for each n.

Definition (Homology)

The *homology* of a chain complex (M_*, d) is the chain complex $(H_*, 0)$, where

$$H^n = Ker(d_n)/Im(d_{n-1}).$$

The Khovanov Chain Complex

We can finally construct the Khovanov Chain Complex:

- Assign to each $\mathcal{I}_\alpha$ of the resolution cube a vector space $V(\mathcal{I}_\alpha)$ of dimension 2^{k_α} over $\mathbb{F}_2$, where k_α is the number of circles.

The Khovanov Chain Complex

We can finally construct the Khovanov Chain Complex:

- Assign to each $\mathcal{I}_\alpha$ of the resolution cube a vector space $V(\mathcal{I}_\alpha)$ of dimension 2^{k_α} over $\mathbb{F}_2$, where k_α is the number of circles.
- We get the sequence

$$\mathcal{C}(\mathcal{D})^i = \bigoplus_{|\mathcal{I}_\alpha|} V(\mathcal{I}_\alpha)$$

with suitable *homological degree* i.

The Khovanov Chain Complex

We can finally construct the Khovanov Chain Complex:

- Assign to each $\mathcal{I}_\alpha$ of the resolution cube a vector space $V(\mathcal{I}_\alpha)$ of dimension 2^{k_α} over $\mathbb{F}_2$, where k_α is the number of circles.
- We get the sequence

$$\mathcal{C}(\mathcal{D})^i = \bigoplus_{|\mathcal{I}_\alpha|} V(\mathcal{I}_\alpha)$$

  with suitable *homological degree* i.
- For each 1-dimensional face, Khovanov defines a linear map.

The Khovanov Chain Complex

We can finally construct the Khovanov Chain Complex:

- Assign to each $\mathcal{I}_\alpha$ of the resolution cube a vector space $V(\mathcal{I}_\alpha)$ of dimension 2^{k_α} over $\mathbb{F}_2$, where k_α is the number of circles.
- We get the sequence

  $$\mathcal{C}(\mathcal{D})^i = \bigoplus_{|\mathcal{I}_\alpha|} V(\mathcal{I}_\alpha)$$

  with suitable *homological degree* i.
- For each 1-dimensional face, Khovanov defines a linear map.
- Direct summing them down columns of the cube give the homomorphisms

  $$d_1^i : \mathcal{C}(\mathcal{D})^i \longrightarrow \mathcal{C}(\mathcal{D})^{i+1}$$

The Khovanov Chain Complex

Lemma (Khovanov)

$d_1^2 = 0$*: This means we can take the homology of the chain complex* $(\mathcal{C}(\mathcal{D})_*, d_1)$*, called the Khovanov Homology of link diagram* $\mathcal{D}$*, or* $Kh(\mathcal{D})$*.*

The Khovanov Chain Complex

Lemma (Khovanov)

$d_1^2 = 0$: *This means we can take the homology of the chain complex* $(\mathcal{C}(\mathcal{D})_*, d_1)$, *called the Khovanov Homology of link diagram* $\mathcal{D}$, *or* $Kh(\mathcal{D})$.

Theorem (Khovanov, Bloom)

Not only is $Kh(\mathcal{D})$ *a link invariant, but it is also invariant under Conway Mutation.*

A Natural Extension

We have a functioning chain complex under 1-dimensional face maps, but what if we consider k dimensions?

A Natural Extension

We have a functioning chain complex under 1-dimensional face maps, but what if we consider k dimensions?

- Zoltan Szabó defines the respective maps $d_2, d_3, ..., d_n$.

A Natural Extension

We have a functioning chain complex under 1-dimensional face maps, but what if we consider k dimensions?

- Zoltan Szabó defines the respective maps $d_2, d_3, ..., d_n$.
- d_2 induces a map d_2^* on the homology $Kh(\mathcal{D})$, and ${d_2^*}^2 = 0$.

The E^k Spectral Page

This means we can take the homology again, giving us $E^2(\mathcal{D})$, i.e. $H(Kh(\mathcal{D}), d_2^*))$.

The E^k Spectral Page

This means we can take the homology again, giving us $E^2(\mathcal{D})$, i.e. $H(Kh(\mathcal{D}), d_2^*))$.

- Inductively we can show that d_k is a differential on $E^{k-1}(\mathcal{D})$ and define $E^k(\mathcal{D}) = H(E^{k-1}(\mathcal{D}), d_k)$. This is called *Szabó's Geometric Spectral Sequence*.

The E^k Spectral Page

This means we can take the homology again, giving us $E^2(\mathcal{D})$, i.e. $H(Kh(\mathcal{D}), d_2^*))$.

- Inductively we can show that d_k is a differential on $E^{k-1}(\mathcal{D})$ and define $E^k(\mathcal{D}) = H(E^{k-1}(\mathcal{D}), d_k)$. This is called *Szabó's Geometric Spectral Sequence*.

Theorem (Szabó)

$E^k(\mathcal{D})$ is a link invariant.

Initial Approaches

The overarching goal is to show that $E^k(\mathcal{D})$ is invariant under Conway Mutation.

Initial Approaches

The overarching goal is to show that $E^k(\mathcal{D})$ is invariant under Conway Mutation.

- We first tried to show that d_k was preserved under mutation, but this is false.

Initial Approaches

The overarching goal is to show that $E^k(\mathcal{D})$ is invariant under Conway Mutation.

- We first tried to show that d_k was preserved under mutation, but this is false.
- We realized there was an alternative approach (adapt Lambert-Cole's proof for $Kh(\mathcal{D})$) more heavily reliant on homological algebra (exact triangles).

Initial Approaches

The overarching goal is to show that $E^k(\mathcal{D})$ is invariant under Conway Mutation.

- We first tried to show that d_k was preserved under mutation, but this is false.
- We realized there was an alternative approach (adapt Lambert-Cole's proof for $Kh(\mathcal{D})$) more heavily reliant on homological algebra (exact triangles).

This requires proving certain properties of a *reduced version* of the complex.

Future Research

The *reduced version* of the Chain Complex comes from choosing a base point on a circle and altering the vector space construction of the original complex.

Future Research

The *reduced version* of the Chain Complex comes from choosing a base point on a circle and altering the vector space construction of the original complex.

- Goal: Prove the reduced version is independent of choice of base point.

Future Research

The *reduced version* of the Chain Complex comes from choosing a base point on a circle and altering the vector space construction of the original complex.

- Goal: Prove the reduced version is independent of choice of base point.
- Use this to prove that $E^2(\mathcal{D})$ is invariant under Conway Mutation.

Future Research

The *reduced version* of the Chain Complex comes from choosing a base point on a circle and altering the vector space construction of the original complex.

- Goal: Prove the reduced version is independent of choice of base point.
- Use this to prove that $E^2(\mathcal{D})$ is invariant under Conway Mutation.
- Generalize to $E^k(\mathcal{D})$.

Acknowledgements

I would very much like to thank the following:

- Dr. Jianfeng Lin, my mentor, for suggesting this project and working extensively with me
- PRIMES-USA, for providing this wonderful opportunity
- Dr. Tanya Khovanova and Dr. Slava Gerovitch
- The MIT Math Department
- My parents